The Brachistochrone Problem

The Brachistochrone (brachistos (greek): short, chronos (greek): time) is the shape of the curve which connects two points on which a ball (a mass point) will require minimal time to get from point A to point B under the assumption of zero friction. Johann Bernoulli solved the problem in 1696. The Brachistochrone problem is a function optimization problem (we are looking for a curve or a trajectory) that directly leads to the Variational Calculus. Let point A have the coordinates (x_A, y_A) and point B (x_B, y_B), the times when the ball passes points A and B are t_A, t_B. We can write for the total time required (denoting the balls speed by v):

$$T = \int_{t_A}^{t_B} dt = \frac{1}{v} ds. \tag{8}$$

Furthermore, conservation of energy ($E_{kin} = E_{pot}$) leads to $v = \sqrt{2gy(x)}$. At the same time, the tangential step ds is given by $ds^2 = dx^2 + dy^2$. Thus, we can write

$$T = \int_{x_A}^{x_B} \frac{1}{\sqrt{2gy(x)}} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \stackrel{!}{=} min. \tag{9}$$

Variational Calculus

We assume that $y(x)$ is a function of x and seek the extremum of the functional

$$I(y) \; = \; \int_{x_0}^{x_1} f\left(x, y, \frac{dy}{dx}\right) dx \; = \; \int_{x_0}^{x_1} f\left(x, y, y'\right) dx \tag{10}$$

Now we assume that $y_0(x)$ is an extremum of $I(y)$, thus $\delta I = 0$ and introduce the function $y_\varepsilon(x) = y_0(x) + \varepsilon h(x)$, where the function $h(x)$ satisfies the following conditions $h(x_0) = h(x_1) = 0$.

$$I(y_\varepsilon) \; = \; F(\varepsilon) \; = \; \int_{x_0}^{x_1} f\left(x, y_\varepsilon, \frac{dy_\varepsilon}{dx}\right) dx \tag{11}$$

Since we assumed that $y_0(x)$ is an extremum of $I(y)$, we know that

$$\delta I \; = \; \left(\frac{dF}{d\varepsilon}\right)_{\varepsilon=0} \; = \; F'(\varepsilon = 0) \; = \; 0. \tag{12}$$

Therefore, we can write

$$
\begin{aligned}
F'(\varepsilon) &= \int_{x_0}^{x_1} \frac{df}{d\varepsilon}\, dx = \int_{x_0}^{x_1} \left(\frac{\partial f}{\partial y_\varepsilon} \frac{\partial y_\varepsilon}{\partial \varepsilon} + \frac{\partial f}{\partial y'_\varepsilon} \frac{\partial y'_\varepsilon}{\partial \varepsilon} \right) dx \\
&= \int_{x_0}^{x_1} \left(\frac{\partial f}{\partial y_\varepsilon} h(x) + \frac{\partial f}{\partial y'_\varepsilon} h'(x) \right) dx. \qquad (13)
\end{aligned}
$$

Partial integration ($\int uv'dx = uv - \int u'v$) of the second term of the integrand and the boundary condition for $h(x)$ lead to

$$
\begin{aligned}
F'(\varepsilon) &= \int_{x_0}^{x_1} \left(\frac{\partial f}{\partial y_\varepsilon} h(x) - \frac{d}{dx} \left(\frac{\partial f}{\partial y'_\varepsilon} \right) h(x) \right) dx \\
&\Rightarrow \\
F'(\varepsilon = 0) &= \int_{x_0}^{x_1} \left(\frac{\partial f}{\partial y_0} h(x) - \frac{d}{dx} \left(\frac{\partial f}{\partial y'_0} \right) h(x) \right) dx \overset{!}{=} 0 \qquad (14)
\end{aligned}
$$

Since equation (14) has to be fulfilled for any $h(x)$, we obtain the Euler-Lagrange differential equation that has to be fulfilled in order that $y_0(x)$ is an extremum of the functional $I(y)$:

$$
\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\left(\frac{\partial y}{\partial x} \right)} \right) = 0. \qquad (15)
$$